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Analytic derivation of dual gluons and monopoles from SU(2) lattice Yang-Mills theory
III. Plaquette representation

Florian Conrady*

Institute for Gravitational Physics and Geometry,
Physics Department, Penn State University,
University Park, PA 16802, U.S.A

Abstract

In this series of three papers, we generalize the derivation of dual photons and monopoles by Polyakov, and Banks, Myerson and Kogut, to obtain gluon-monopole representations of SU(2) lattice gauge theory. Our approach is based on semiclassical weak-coupling expansions.

In this third article, we start from the plaquette representation of 3-dimensional SU(2) lattice gauge theory. By extending a work of Borisenko, Voloshin and Faber, we transform the expectation value of a Wilson loop into a path integral over a dual gluon field and monopole variables. The action contains the tree-level Coulomb interaction and a nonlinear coupling between dual gluons, monopoles and current.

By making an additional assumption on the monopole self-energy, we can generalize Polyakov's derivation of confinement to gauge group SU(2) in 3 dimensions.

1 Introduction

In the analysis of lattice gauge theories, it proves very useful to transform between different representations of the same theory. In the 70's, Banks, Myerson, and Kogut have shown that U(1) lattice gauge theory can be transformed exactly into a representation by a dual photon field and monopoles [1]. This photon-monopole representation was derived earlier by a different method by Polyakov [2, 3].

One can demonstrate with it that electrostatic charges are confined in 3 dimensions [3, 4], and that there is a phase transition in 4 dimension [1, 5, 6]: in $d = 3$ the monopoles condense along a string between the charges and create a linear confining potential. In dimension 4 there is a critical size of the coupling at which such a condensation sets in.

The example of U(1) fostered hopes that one could generalize this scheme to non-abelian gauge groups: it led to the conjecture that confinement in U(1) and SU(N) result from the same mechanism, and that it has an analogy in dual type II superconductors [7, 8].

*email: conrady at gravity dot psu dot edu

The problem, however, proves to be much harder than in the abelian case. Presently, we only know of three non-abelian analogues of U(1) lattice representations: apart from the defining path integral, we have 1. a first-order representation, which can be viewed as a lattice version of BF Yang-Mills theory, 2. the spin foam representation [9, 10, 11, 12], and 3. the plaquette representation [13, 14, 15]. The first two representations are defined for any dimension $d \geq 2$, while the latter has been only constructed in 3 dimensions so far.

It is difficult to go beyond these three representations: both the plaquette and the spin foam representation are governed by non-abelian generalizations of constraints that we find in the abelian case. For U(1) these constraints can be solved exactly, so that one can move on to obtain the dual $\mathbb{Z}$ gauge theory and the photon-monopole representation. In the non-abelian case, we do not know at present, how one could solve these constraints exactly, or if there is at all a meaningful sense in which representations beyond the known ones exist.

This article is the third in a series of three papers, where we derive non-abelian analogues of the photon-monopole representation for gauge group SU(2). We are not able to do this by exact transformations, and have to rely instead on semiclassical weak-coupling expansions. The expansions lead us to three versions of a gluon-monopole representation of SU(2) lattice gauge theory. In this paper, our starting point is the plaquette representation in 3 dimensions, as it was given by Borisenko, Voloshin and Faber[1]: in their paper, the expectation value of a Wilson loop is transformed into a perturbation series, where the generating functional contains a continuous and discrete variable. The variables resemble the dual photon and monopole variables of U(1), and we interpret them as a dual gluon and monopole field. To lowest order, this representation does not include any contribution from the Wilson loop.

We extend the work of Borisenko et al. by incorporating a lowest-order contribution from the Wilson loop. This is done with the help of the Kirillov trace formula. In this way, we arrive at a lowest-order gluon-monopole representation that has a coupling to a source current J. The structure of the resulting action stands in close analogy to that for U(1): it reproduces roughly the tree-level Coulomb interaction, and the coupling between monopoles, dual gluons and current is similar to that of the abelian case. There is an important difference, however: it consists in the fact that the dual gluon field is $\mathrm{su}(2) \simeq \mathbb{R}^3$-valued (and not $\mathbb{R}$-valued) and that the monopoles couple to the *length* of field vectors. As a result, the gluon-monopole coupling is nonlinear.

The similarity to U(1) naturally suggests that one could try to generalize Polyakov's method of deriving confinement. We propose a way to do so, but it requires an additional heuristic assumption: since we are not able to integrate over the dual gluon field as it was done for U(1), we have to *assume* a renormalization group step that generates a monopole self-energy at a scale M below the cutoff scale. With that assumption, the rest follows as in Polyakov's paper and one arrives at a non-vanishing string tension for the Wilson loop in the representation $j = 1/2$.

In all three papers, the derivations involve semiclassical weak-coupling expansions. It remains to be checked if these semiclassical methods lead to reliable approximations, as in Polyakov's work for U(1) and $d = 3$, or if there occur problems due to higher-order corrections.

The paper is organized as follows: in section 2, we review Polyakov's derivation of confinement for U(1) in 3 dimensions. The definition of the plaquette representation is given in sec. 3. In sec. 4, we derive the representation by dual gluons and monopoles. The tentative derivation of the area law follows in sec. 5. The results are summarized and discussed in the

[1] It is related to earlier plaquette (or field strength) representations by Halpern and Batrouni [16, 13, 14].

final section.

Notation and conventions

κ denotes a d-dimensional hypercubic lattice of side length L with periodic boundary conditions. The lattice constant is a. Depending on the context, we use abstract or index notation to denote oriented cells of κ: in the abstract notation, vertices, edges, faces and cubes are written as v, e, f and c respectively. In the index notation, we write x, $(x\mu)$, $(x\mu\nu)$, $(x\mu\nu\rho)$ etc. Correspondingly, we have two notations for chains. Since the lattice is finite, we can identify chains and cochains. As usual, ∂, d and $*$ designate the boundary, coboundary and Hodge dual operator respectively. Forward and backward derivative are defined by

$$\nabla_\mu f_x = \frac{1}{a}\left(f_{x+a\hat{\mu}} - f_x\right)\,, \qquad \overline{\nabla}_\mu f_x = \frac{1}{a}\left(f_x - f_{x-a\hat{\mu}}\right) \tag{1}$$

where $\hat{\mu}$ is the unit vector in the μ-direction. The lattice Laplacian reads

$$\Delta = \overline{\nabla}_\mu \nabla_\mu\,. \tag{2}$$

For a given unit vector $u = \hat{\mu}$ and a 1-chain $J_{x\mu}$, we define

$$(u\cdot\nabla)^{-1} J_{x\mu} := \sum_{x'_\mu \leq x_\mu} J_{(x_1,\ldots,x'_\mu,\ldots,x_d)\,\mu}\,. \tag{3}$$

We use units in which $\hbar = c = 1$ and $a = 1$. For some quantities, the a-dependence is indicated explicitly.

2 Confinement in 3-dimensional U(1) lattice gauge theory

In this paper we attempt to generalize the derivation of the photon-monopole representation from U(1) to SU(2) by starting from the plaquette representation. The result will be similar, but not identical, to what we got in paper I and II from the BF Yang-Mills and the spin foam representation. Among the three actions obtained in paper I, II and III, the one we derive here will be most similar to that of the abelian case. The analogy is, in fact, so close that it immediately suggests to go one step further: namely, to generalize also the derivation of the confining potential, as it was given for U(1) and $d = 3$ by Polyakov [3] and Banks, Myerson and Kogut [1].

We will describe this generalization in section 5. As a preparation for that, we review the derivation for U(1) in the present section. We also go into some technical details that will play a role when making the transition to SU(2).

Let κ be the 3-dimensional lattice. For a Wilson loop C in the representation $q \in \mathbb{Z}$, the transformation to the photon-monopole representation yields the following lattice path integral:

$$\begin{aligned}\langle \mathrm{tr}_q W_C\rangle = \frac{1}{Z}\int\limits_{\mathbb{R}^3}\left(\prod_x \mathrm{d}^3\varphi_x\right)\sum_{m_x}\exp\Bigg[\sum_x\Bigg(&\frac{1}{2\beta}\,\varphi_x\Delta\varphi_x + 2\pi\mathrm{i}\left(\varphi_x + \Delta^{-1}\nabla_\mu \overline{b}_{x\mu}\right) m_x \\ &+ \frac{1}{2\beta}\,J_{x\mu}\Delta^{-1}_{xy}J_{y\mu}\Bigg)\Bigg]\end{aligned} \tag{4}$$

Let us explain the notation: W_C denotes the holonomy along the Wilson loop C in the fundamental representation. The loop and the charge q determine a current

$$J_{x\mu} = q\, C_{x\mu}\,. \tag{5}$$

The inverse temperature β is related to the gauge coupling via

$$\beta = \frac{1}{ag^2}\,. \tag{6}$$

The 1-chain $\overline{b}_{x\mu}$ is an abbreviation for

$$\overline{b}_{x\rho} = -\epsilon_{\rho\mu\nu} u_\mu\, (u\cdot\nabla)^{-1} J_{x\nu}\,. \tag{7}$$

We keep the notation of paper I, where we regard $\overline{b}$ as a particular solution of the b-field in the BF Yang-Mills representation. We abbreviate

$$\Delta^{-1}\nabla_\mu \overline{b}_{x\mu} := \sum_y \Delta^{-1}_{xy}\nabla_\mu \overline{b}_{y\mu}\,. \tag{8}$$

Zero momentum modes of φ are excluded in the path integral.

Observe that in (4) we have already rewritten the exponent in such a way that the Coulomb interaction between the currents is factored off. This can be done without performing any integration over φ.

On the lattice there are essentially two ways to obtain the representation (4) from the original lattice gauge theory: either we integrate out the connection and solve the resulting constraint (along the lines of Banks et al.), or we start from the plaquette representation, expand the constraint delta in basis functions and integrate out the plaquette variables. The latter method is explained in sec. 3.1 of [15]. Both derivations yield the same result, apart from perturbative corrections that arise from different choices of the plaquette action

By performing the Gaussian integration over the field φ, we arrive at the description in terms of a Coulomb gas of monopoles:

$$\langle \mathrm{tr}_q W_C\rangle = \sum_{m_x} \exp\left[\sum_{xy}\left(2\pi^2\beta\, m_x\Delta^{-1}_{xy} m_y + 2\pi \mathrm{i}\, m_x \Delta^{-1}_{xy}\nabla_\mu \overline{b}_{y\mu} + \frac{1}{2\beta}\, J_{x\mu}\Delta^{-1}_{xy} J_{y\mu}\right)\right] \tag{9}$$

The quantity $\nabla_\mu \overline{b}_{x\mu}$ can be interpreted as a fixed dipole sheet of magnetic charges along a surface bounded by the Wilson loop: it creates a magnetostatic potential $-\Delta^{-1}_{xy}\nabla_\mu\overline{b}_{y\mu}$ for the monopole gas.

We now follow the appendix of Banks et al. [1]—together with clarifying inputs from references [17] and [4]—to obtain Polyakov's equations of confinement. We begin by splitting the Coulomb potential of the monopoles into a regularized potential and a Yukawa potential for a mass M:

$$-\Delta^{-1} = -\tilde{\Delta}^{-1} + \left(-\Delta + M^2\right)^{-1}\,, \qquad \text{where} \quad -\tilde{\Delta}^{-1} := -\Delta^{-1} - \left(-\Delta + M^2\right)^{-1}\,. \tag{10}$$

The regularized potential $-\tilde{\Delta}^{-1}$ accounts for the energies at monopole-monopole distances larger than $1/M$, while the Yukawa potential gives the energy at distances smaller than $1/M$.

If the typical distance between monopoles is much larger than $1/M$, the Yukawa term is essentially the self-energy of the monopoles and we can approximate it by

$$\begin{aligned} \sum_{xy} m_x \left(-\Delta + M^2\right)^{-1}_{xy} m_y &\approx \left(-\Delta + M^2\right)^{-1}_{00} \sum_x m_x^2 \\ &\equiv v_0 \sum_x m_x^2 \,. \end{aligned} \tag{11}$$

Thus, the splitting (10) amounts to an extraction of the monopole self-energy from the total monopole energy. By reintroducing the field φ *after* this extraction, we get

$$\begin{aligned} \langle \mathrm{tr}_q W_C \rangle = \frac{1}{Z} \int\limits_{\mathbb{R}^3} \left(\prod_x \mathrm{d}^3\varphi_x\right) \sum_{m_x} \exp\Bigg[\sum_x \Bigg(\frac{1}{2\beta}\,\varphi_x \tilde{\Delta} \varphi_x - 2\pi^2 \beta v_0\, m_x^2 + 2\pi\mathrm{i}\left(\varphi_x + \Delta^{-1}\nabla_\mu \bar{b}_{x\mu}\right) m_x \\ + \frac{1}{2\beta}\, J_{x\mu} \Delta^{-1}_{xy} J_{y\mu}\Bigg)\Bigg] \,. \end{aligned} \tag{12}$$

The only difference between equations (4) and (12) is the appearance of the regulated Laplace operator and the self-energy term for the monopoles. We can view this as the result of a renormalization group step that

1. replaces the original field φ at the cutoff scale $1/a$ by an effective field φ with cutoff at M, and

2. generates a self-energy for the monopoles.

Suppose for a moment that we had not known how to compute the intermediate formula (9) in which the splitting (10) of energies is performed. In that case, we could have tried to *guess* the transition from (4) to (12) by *assuming* that a renormalization leads to a regularized Laplacian and a self-energy of the monopoles.

This is the situation we will face when coming to the non-abelian theory: in that case expression (4) will be replaced by a non-Gaussian integral over a field φ, and we do not know how to compute the counterpart of formula (9). Instead we will try to guess the answer by assuming a renormalization that brings us directly from the non-abelian analogue of (4) to the analogue of (12). This will be the only heuristic or adhoc input in our derivation of the quark potential, and further analysis has to tell whether this step is permissible or not.

In the computation of the U(1) potential, the transition from (4) and (12) is crucial, as it makes the dampening of monopole excitations explicit. From here on one can follow Polyakov's arguments to derive the area law [3]: let us abbreviate

$$\eta_x = 2\pi \Delta^{-1} \nabla_\mu \bar{b}_{x\mu} \,, \tag{13}$$

and apply a rescaling and shift on φ:

$$2\pi\varphi + \eta \quad \rightarrow \quad \varphi \tag{14}$$

We also write

$$V_{JJ} = -\frac{1}{2\beta}\, J_{x\mu} \Delta^{-1}_{xy} J_{y\mu} \tag{15}$$

for the Coulomb interaction between the currents:

$$\langle \mathrm{tr}_q W_C \rangle = \frac{1}{Z} \int\limits_{\mathbb{R}^3} \left(\prod_x \mathrm{d}^3\varphi_x \right) \sum_{m_x} \mathrm{e}^{-V_{JJ}} \\ \times \exp\left[\sum_x \left(\frac{1}{8\pi^2\beta} (\varphi_x - \eta_x)\tilde{\Delta}(\varphi_x - \eta_x) - 2\pi^2\beta\gamma\, m_x^2 + \mathrm{i}\, m_x\varphi_x \right) \right] \tag{16}$$

In the continuum limit $2\pi^2\beta v_0$ is large, and monopole excitations are strongly suppressed. We therefore apply the dilute gas approximation and restrict the sum over monopoles:

$$\sum_{m_x \in \mathbb{Z}} \quad \rightarrow \quad \sum_{m_x = 0, \pm 1} . \tag{17}$$

Thus, we have

$$\begin{aligned} \sum_{m_x} \exp\left[-2\pi^2\beta v_0\, m_x^2 + \mathrm{i}\, m_x\varphi_x \right] &\approx 1 + 2\exp\left(-2\pi^2\beta\gamma \right) \cos\varphi_x \\ &\approx \exp\left[2\exp\left(-2\pi^2\beta v_0 \right) \cos\varphi_x \right] , \end{aligned} \tag{18}$$

and the path integral becomes

$$\langle \mathrm{tr}_q W_C \rangle = \frac{1}{Z} \int\limits_{\mathbb{R}^3} \left(\prod_x \mathrm{d}^3\varphi_x \right) \mathrm{e}^{-V_{JJ}} \exp\left[\sum_x \left(\frac{1}{8\pi^2\beta} (\varphi_x - \eta_x)\tilde{\Delta}(\varphi_x - \eta_x) + 2\,\mathrm{e}^{-2\pi^2\beta v_0} \cos\varphi_x \right) \right] . \tag{19}$$

Up to a shift in the energy and the regulator in Δ, the exponent is the action of sine-Gordon theory on the lattice. The mass of elementary excitations is given by

$$M_D^2 = 8\pi^2\beta\, \mathrm{e}^{-2\pi^2\beta v_0} . \tag{20}$$

In the context of plasma physics, this is called the Debye mass.

The next step is a saddle point approximation. The saddle points of (19) are determined by the nonlinear Debye equation

$$-\tilde{\Delta}(\varphi - \eta) = M_D^2 \sin\varphi . \tag{21}$$

We approximate this further by replacing the regulated Laplacian by the full Laplacian. We then get

$$\Delta\varphi = 2\pi\nabla_\mu \overline{b}_{x\mu} - M_D^2 \sin\varphi \tag{22}$$

where we plugged in the definition (13).

For simplicity we assume that the Wilson loop C bounds a rectangular surface S in the upper-right 1-2-plane with anti-clockwise orientation. By setting $u = \hat{1}$ in (7) we find that $\overline{b}_{x\mu}$ is the normal vector to this surface S, i.e.

$$b_{x\rho} = \epsilon_{\rho\mu\nu} S_{x\mu\nu} . \tag{23}$$

If we view eq. (22) as the equation for a magnetostatic potential φ, the first term on the right-hand side of (22) corresponds to a dipole sheet of charges 2π along S:

$$-2\pi\nabla_\mu \overline{b}_{x\mu} = 2\pi \left(\delta_{x3,0} - \delta_{x3,-a} \right) \tag{24}$$

This implies that the potential φ must jump by 2π when we cross the surface from $x_3 < 0$ to $x_3 > 0$.

For the region above and below the surface, we approximate equation (22) by the one-dimensional continuum equation

$$\frac{\partial^2\varphi}{\partial x_3^2} = -M_D^2 \sin\varphi\,. \tag{25}$$

This is the equation for domain walls of the sine-Gordon theory in 1+1 dimensions. One can choose parts of such domain wall solutions for $x_3 < 0$ and $x_3 > 0$ respectively, and fit them together, so that they produce the needed discontinuity at S. The resulting solution in 3 dimensions is

$$\varphi(x) \quad \approx \quad \begin{cases} 4\,\mathrm{sgn}(x_3)\arctan\left(\mathrm{e}^{-M_D|x_3|}\right)\,, & (x_1,x_2)\in[0,T]\times[0,R]\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{26}$$

We plug this solution back into (19) and use the trivial saddle point for Z, so $Z = 1$. It is clear that the action of this solution equals $A\mathcal{S}_{\mathrm{dw}}$, where A is the area of the surface S and $\mathcal{S}_{\mathrm{dw}}$ is the action (or minus the energy) of a domain wall in the 1+1-dimensional theory: the explicit value is

$$\mathcal{S}_{\mathrm{dw}} \approx \frac{2M_D}{\pi^2\beta}\,. \tag{27}$$

Thus, we obtain

$$\langle \mathrm{tr}_1 W_C\rangle \approx \exp\left(-V_{JJ} - \sigma A\right) \tag{28}$$

with a string tension

$$\sigma = \frac{2M_D}{\pi^2\beta}\,. \tag{29}$$

This shows that the Wilson loop obeys an area law. The dependence on the energy scale M (in eq. (10)) is discussed in Duncan's and Mawhinney's paper [17].

3 Plaquette representation of 3-dimensional SU(2) lattice Yang-Mills theory

The basic idea behind the plaquette (or field-strength) formulation is a change of variables from holonomies along edges to holonomies around plaquettes. The path integral over the new plaquette variables is constrained, since plaquette holonomies have to satisfy Bianchi constraints: the well-known Bianchi identity of abelian lattice gauge theory, or generalizations thereof for non-abelian gauge theories.

There exist different schemes for constructing such plaquette representations (see [13, 14, 15]). In this paper, we will use the formulation of Borisenko, Voloshin and Faber for SU(N) lattice gauge theory in 3 dimensions [15]. Although the underlying idea is simple, a precise description of this representation is quite involved, as it requires numerous conventions on orientations and orderings. For that reason, we will review the essential definitions in this section.

To simplify the presentation, we will ignore all boundary effects. In the next section, we will restrict ourselves to the first term in a weak-coupling expansion, and to that order any

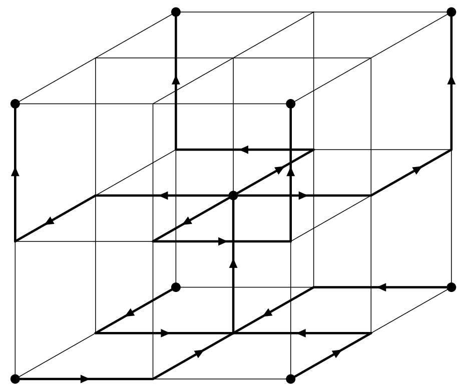

Figure 1: Eight cubes around an even point: the thick lines indicate a possible choice of connectors.

boundary-related modifications drop out. The proper treatment of boundary conditions is given in [15].

Edge (or link) variables are denoted by U_e, and W_f designates the holonomy around a face f:

$$W_f = \prod_{e\subset\partial f} U_e \tag{30}$$

It is assumed that we have chosen some starting point $v \subset f$, so that W_f is the product of edge holonomies U_e, starting at v, and following the orientation of the face f. Whenever we have such (or similar) products of group elements, we use left multiplication, i.e.

$$\prod_{i=0}^{n} U_i = U_n U_{n-1} \ldots U_2 U_1 \,. \tag{31}$$

The convention for starting points will be fixed further below.

In the standard formulation of lattice gauge theory, the expectation value of a Wilson loop C is defined by the path integral

$$\langle \mathrm{tr}_j W_C \rangle = \int \left(\prod_{e\subset\kappa} \mathrm{d}U_e \right) \exp\left[-\sum_{f\subset\kappa} \frac{\beta}{4}\,\mathrm{tr}\left(W_f + W_f^{-1} \right) \right] \chi_j\left(W_C\right)\,. \tag{32}$$

Here, the exponent is given by the Wilson action and

$$\beta = \frac{4}{ag^2}\,. \tag{33}$$

W_C stands for the holonomy around the Wilson loop C, where, again, a starting point along C is assumed. $\chi_j = \mathrm{tr}_j$ is the character in the representation j. It is understood that for each pair of edge orientations e, e^{-1}, we integrate only over one edge variable U_e, with the other one being fixed by

$$U_{e^{-1}} = U_e^{-1}\,. \tag{34}$$

The plaquette representation arises from a change of variables: from the edge (or link) variables U_e to the W_f's, which we call face (or plaquette) variables. This change of variables is achieved in four steps:

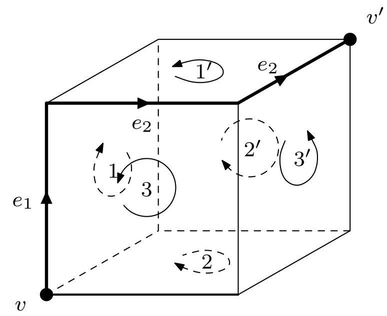

Figure 2: Cube of κ with chosen connector.

1. Introduce integrations over plaquette variables W_f + delta constraints (30) on them.
2. Reexpress some of these constraints as non-abelian Bianchi identities.
3. Impose the maximal axial gauge.
4. Integrate out the edge variables U_e.

The result is a path integral over plaquette variables where of all the constraints on W_f only one type is left, namely, the non-abelian Bianchi identities.

To write down the non-abelian Bianchi identities we need to introduce a number of conventions: consider the set of vertices v whose coordinates (x_1, x_2, x_3) are all even and call them the even vertices. Likewise, we call the vertices with odd coordinates odd. Clearly, each cube of κ contains an even and odd point as corners, and each face of κ contains either one even or one odd vertex. We will use this to specify our convention for starting points of face holonomies W_f: when associating a holonomy to a face we will always take the even or odd point in it as the starting point.

In every cube of κ we choose an oriented path consisting of three edges that connect the even and odd point. We call this path a connector. We can choose the connectors in such a way that there are altogether four types of connectors on the lattice (see Fig. 1). To each connector we associate a holonomy

$$K_{v'v} = U_{e_3} U_{e_2} U_{e_1} \,. \tag{35}$$

Let us now specify the non-abelian Bianchi identities. Consider one cube c as in Fig. 2. Take the boundary ∂c and choose an orientation for it. This will induce an orientation for each face $f \subset \partial c$. Call the starting point of the connector v and the end point v'. We have three faces of the cube that meet at v: order them f_1, f_2, f_3, starting with the face whose orientation agrees with that of the connector, then the one which does not intersect with the connector, and then the remaining one (whose orientation is opposite to that of the connector). Likewise, we have three faces meeting at v', and we order them f'_1, f'_2, f'_3, according to the same rule as before (see Fig. 2).

With these conventions the non-abelian Bianchi identity looks as follows:

$$V_c = K^{-1}_{p'p}\, W_{f'_3} W_{f'_2} W_{f'_1}\, K_{p'p}\, W_{f_3} W_{f_2} W_{f_1} = \mathbb{1} \tag{36}$$

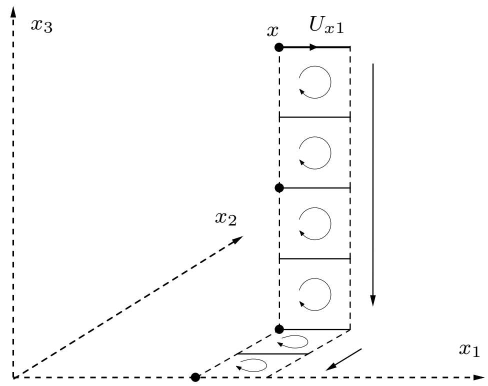

Figure 3: Edge variables as products of plaquette variables: in the maximal axial gauge the edge variable U_{x1} can be expressed as a product of plaquette variables. The dots represent even points, and the arrows indicate the order of the product. All edge variables on dashed lines equal $\mathbb{1}$ due to the maximal axial gauge.

W_{f_i} denotes the holonomy around the face f_i with the starting point at p, and likewise the starting point for $W_{f_i'}$ is taken to be p'. The reader may convince himself by a drawing that this identity is indeed satisfied.

In the abelian case, the connectors would cancel and the expression would reduce to the usual abelian Bianchi identity. Here, this does not happen. At first sight one might think that this makes a formulation in terms of plaquette variables impossible, since the appearance of connectors prevents us from eliminating the edge variables completely. It turns out, however, that by the steps 3 and 4 above, the edge variables appearing in (36) will be expressed in terms of plaquette variables, and thus one ends up with a path integral that contains only plaquette variables plus constraints on them.

The expressions for the edge variables depend on how the edge is situated relative to the origin and the even and odd points: for example, if x is an even point and $x_i > 0$, the edge variable U_{x1} is given by (see Fig. 3)

$$U_{(x_1,x_2,x_3),1} = \prod_{x_2'=x_2-1}^{0} W_{(x_1,x_2',0),21} \prod_{x_3'=x_3-1}^{0} W_{(x_1,x_2,x_3'),31} \,. \tag{37}$$

Similar equations hold for other edge variables. To write them down explicitly, we would have to distinguish many different cases, although the principle is the same as in equation (37). In this paper, these details do not matter, since we use a weak-coupling expansion to lowest order, and the formulas for the connectors and edge variables drop out.

In a similar fashion the Wilson loop can be expressed as a product of plaquette variables. We take C to be a rectangular loop in the 1-2-plane with anti-clockwise orientation, where the lower-left and upper-right corner are given by even points $(0,0,0)$ and $(T,R,0)$, $T,R > 0$. Then, the holonomy around C with starting point $x = (0,0,0)$ equals

$$W_C = \prod_{x_1=T-1}^{0} \prod_{x_2=R-a}^{0} W_{(x_1,x_2,0),12} \,, \tag{38}$$

i.e. it is an ordered product of plaquette variables whose plaquettes fill the surface enclosed by the Wilson loop.

By going through the steps 1 to 4 and using the formulas (36), (37) and (38), the original path integral (32) is rewritten as a constrained path integral over plaquette variables W_f:

$$\langle \mathrm{tr}_j W_C \rangle = \int \left(\prod_{f \subset \kappa} \mathrm{d}W_f \right) \left(\prod_{c \subset \kappa} \delta\left(V_c\right) \right) \exp\left[\sum_f \frac{\beta}{4} \mathrm{tr}\left(W_f + W_f^{-1} \right) \right] \chi_j\left(W_C\right) \tag{39}$$

If we change to an index notation, writing $(x\mu)$ instead of e and $(x\mu\nu)$ in place of f, the same reads

$$\langle \mathrm{tr}_j W_C \rangle = \int \left(\prod_{x\mu\nu} \mathrm{d}W_{x\mu\nu} \right) \left(\prod_{x} \delta\left(V_x\right) \right) \exp\left[\sum_x \sum_{\mu<\nu} \frac{\beta}{4} \mathrm{tr}\left(W_{x\mu\nu} + W_{x\mu\nu}^{-1} \right) \right] \chi_j\left(W_C\right) . \tag{40}$$

4 Representation as dual gluons and monopoles

After deriving their plaquette representation, Borisenko et al. continue with the construction of a weak-coupling (large β, low-temperature) perturbation theory. The associated generating functional can be considered as a partition function of dual gluons and monopoles of SU(2) lattice gauge theory, and generalizes the photon-monopole partition function of U(1).

What we add to this scheme is the following: we use the Kirillov trace formula [18], as in paper I, to incorporate the Wilson loop in the exponent. As a result, the lowest-order contribution of the Wilson loop does not result from an exponentiation of diagrams, as in usual perturbation theory [19, 20, 21], but appears there from the very beginning. It may be possible to derive this also from the perturbation theory of [15]. This has not been shown so far, however. With this step we find a generalization of the photon-monopole action that does not only contain the dual gluons and monopoles, but also the lowest-order contribution of the Wilson loop. This incorporation of the loop current will be crucial in section 5 where we propose a non-abelian generalization of Polyakov's derivation of the area law.

We will now describe the steps that lead to the description in terms of dual gluons and monopoles: first we will rewrite measure, action and Bianchi constraint in terms of Lie algebra elements. To do that for the Bianchi constraint, the delta function is expressed as a sum over characters, and the character, by the Kirillov trace formula, as a function of a Lie algebra element. We apply the same formula to the trace over the Wilson loop, so that it is parametrized by a Lie algebra element.

In the second step, we apply the Poisson summation formula which trades the discrete sum over representations for new degrees of freedom that can be regarded as (dual) gluons and monopoles. The third step consists in the semiclassical expansion, of which we only keep the lowest order.

Kirillov trace and Poisson summation formula

We start by rewriting everything in terms of Lie algebra elements: if we parametrize the plaquette variables as

$$W_{x\mu\nu} = \mathrm{e}^{\mathrm{i}\,\omega^a_{x\mu\nu}\sigma^a/2} , \qquad |\omega_{x\mu\nu}| < 2\pi , \tag{41}$$

the action becomes

$$\frac{\beta}{4}\,\mathrm{tr}\left(W_{x\mu\nu}+W_{x\mu\nu}^{-1}\right)=\frac{\beta}{2}\,\cos\left(|\omega_{x\mu\nu}|/2\right)\,. \tag{42}$$

The measure takes the form

$$\int \mathrm{d}W_{x\mu\nu}\;\ldots\quad=\quad\frac{1}{\pi^2}\int\limits_{B_{2\pi}(0)}\mathrm{d}^3\omega_{x\mu\nu}\;\frac{\sin^2\left(|\omega_{x\mu\nu}|/2\right)}{\left(|\omega_{x\mu\nu}|/2\right)^2}\;\ldots\,. \tag{43}$$

The Bianchi constraint is written as

$$\delta\left(V_x\right)=\sum_j\left(2j+1\right)\chi_j(V_x)\,. \tag{44}$$

Let $\upsilon_x^a\sigma^a/2$ be the Lie algebra element associated to the constraint V_x, i.e.

$$V_x=\mathrm{e}^{\mathrm{i}\,\upsilon_x^a\sigma^a/2} \tag{45}$$

Then the Kirillov trace formula gives

$$\chi_j\left(V_x\right)=\frac{(2j+1)|\upsilon_x|/2}{4\pi\sin(|\upsilon_x|/2)}\int_{S^2}\mathrm{d}n\;\;\mathrm{e}^{\mathrm{i}\,(2j+1)\,n\cdot\upsilon_x/2}\,. \tag{46}$$

The integral runs over unit vectors n in $\mathbb{R}^3$, i.e. over the 2-sphere. Next we use the Poisson summation formula to replace the sum over representations by an integral *and* a sum:

$$\begin{aligned}\sum_j\,(2j+1)\chi_j(V_x)\quad&=\quad\sum_{k\in\mathbb{N}_0}\frac{k^2|\upsilon_x|/2}{4\pi\sin(|\upsilon_x|/2)}\int_{S^2}\mathrm{d}n\;\;\mathrm{e}^{\mathrm{i}\,k\,n\cdot\upsilon_x/2}\\&=\quad\frac{1}{2}\sum_{k\in\mathbb{Z}}\frac{k^2|\upsilon_x|/2}{4\pi\sin(|\upsilon_x|/2)}\int_{S^2}\mathrm{d}n\;\;\mathrm{e}^{\mathrm{i}\,k\,n\cdot\upsilon_x/2}\\&=\quad\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}r\,r^2\sum_{m\in\mathbb{Z}}\frac{|\upsilon_x|/2}{4\pi\sin(|\upsilon_x|/2)}\int_{S^2}\mathrm{d}n\;\;\mathrm{e}^{\mathrm{i}\,r\,n\cdot\upsilon_x/2\,+\,2\pi\mathrm{i}\,r\,m}\\&=\quad\frac{1}{8\pi}\int\limits_{\mathbb{R}^3}\mathrm{d}^3\varphi\;\sum_{m\in\mathbb{Z}}\frac{|\upsilon_x|/2}{\sin(|\upsilon_x|/2)}\,\mathrm{e}^{\mathrm{i}\,\varphi\cdot\upsilon_x\,+\,4\pi\mathrm{i}\,|\varphi|m}\end{aligned} \tag{47}$$

In the last equation we combined the integral over the radius and the unit vector into an integral over $\mathbb{R}^3$.

To deal with the Wilson loop, we apply the Kirillov trace formula a second time: by writing

$$W_C=\mathrm{e}^{\mathrm{i}\,\omega_C^a\sigma^a/2} \tag{48}$$

we obtain

$$\chi_j\left(W_C\right)=\frac{(2j+1)|\omega_C|/2}{4\pi\sin(|\omega_C|/2)}\int_{S^2}\mathrm{d}n\;\;\mathrm{e}^{\mathrm{i}\,(2j+1)\,n\cdot\omega_C/2}\,. \tag{49}$$

With all this the path integral becomes

$$
\begin{aligned}
\langle \mathrm{tr}_j W_C \rangle = \frac{1}{Z} \int\limits_{B_{2\pi}(0)} \left(\prod_{x\mu\nu} \mathrm{d}^3\omega_{x\mu\nu} \right) \int\limits_{\mathbb{R}^3} \left(\prod_x \mathrm{d}^3\varphi_x \right) \sum_{m_x \in \mathbb{Z}} \int\limits_{S^2} \mathrm{d}n \\
\times \left(\prod_x \frac{|\upsilon_x|/2}{\sin(|\upsilon_x|/2)} \right) \frac{(2j+1)|\omega_C|/2}{4\pi \sin(|\omega_C|/2)} \\
\times \exp \left[\sum_x \left(\sum_{\mu<\nu} \beta \cos\Big(|\omega_{x\mu\nu}|/2 \Big) + \mathrm{i}\,(2j+1)\, n\cdot\omega_C/2 + \mathrm{i}\,\varphi_x \cdot \upsilon_x + 4\pi\mathrm{i}\,|\varphi_x|\, m_x \right) \right]
\end{aligned}
\tag{50}
$$

The integration range of $\omega_{x\mu\nu}$ is the ball of radius 2π in $\mathbb{R}^3$. The scalar fields φ and m are $\mathbb{R}^3$- and $\mathbb{Z}$-valued respectively. Field-independent constants that appear both in the numerator and in Z were dropped.

Semiclassical expansion

Let us expand all quantities in (50) in powers of $\omega_{x\mu\nu}$: the Wilson action yields

$$
\beta \cos\Big(|\omega_{x\mu\nu}|/2 \Big) = \beta \left(1 - \frac{1}{16}\, \omega^2_{x\mu\nu} + o(\omega^3_{x\mu\nu}) \right) . \tag{51}
$$

When we expand the non-abelian Bianchi identity (36), the contribution of the connectors cancel to lowest order, and

$$
\upsilon_x = \frac{1}{2}\, \epsilon_{\rho\mu\nu} \nabla_\rho \omega_{x\mu\nu} + o(\omega^2_{x\mu\nu}) \, . \tag{52}
$$

It follows from equation (38) that

$$
\omega_C = \sum_x \frac{1}{2}\, S_{x\mu\nu}\omega_{x\mu\nu} + o(\omega^2_{x\mu\nu}) \, , \tag{53}
$$

where S is the minimal surface spanned by the Wilson loop C. From the measure factors we obtain

$$
\frac{|\omega_{x\mu\nu}|/2}{\sin(|\omega_{x\mu\nu}|/2)} = \exp \left(\frac{1}{6} \left(\frac{|\omega_{x\mu\nu}|}{2} \right)^2 + \frac{1}{180} \left(\frac{|\omega_{x\mu\nu}|}{2} \right)^4 + \dots \right) . \tag{54}
$$

$$
\frac{|\upsilon_x|/2}{\sin(|\upsilon_x|/2)} = \exp \left(\frac{1}{6} \left(\frac{|\frac{1}{2}\epsilon_{\rho\mu\nu} \nabla_\rho \omega_{x\mu\nu} + \dots|}{2} \right)^2 + \dots \right) . \tag{55}
$$

$$
\frac{|\omega_C|/2}{\sin(|\omega_C|/2)} = \exp \left(\frac{1}{6} \left(\frac{|\sum_x \frac{1}{2}\, S_{x\mu\nu}\omega_{x\mu\nu} + \dots|}{2} \right)^2 + \dots \right) . \tag{56}
$$

Each of the terms (51)–(56) contributes with linear, quadratic and higher orders of $\omega_{x\mu\nu}$.

We consider the continuum limit in which $\beta \gg 1$ and retain only the lowest order in the semiclassical expansion around $\omega_{x\mu\nu} = 0$: the quadratic term in (51) is much larger than the quadratic terms coming from (52)–(56), so we keep only the former one. We will also drop *all* orders that are higher than quadratic in $\omega_{x\mu\nu}$. This yields the following semiclassical approximation[2]:

$$\langle \mathrm{tr}_j W_C \rangle = \frac{1}{Z} \int\limits_{B_{2\pi}(0)} \Bigg(\prod_{x\mu\nu} \mathrm{d}^3\omega_{x\mu\nu} \Bigg) \int\limits_{\mathbb{R}^3} \Bigg(\prod_x \mathrm{d}^3\varphi_x \Bigg) \sum_{m_x \in \mathbb{Z}} \frac{2j+1}{4\pi} \int\limits_{S^2} \mathrm{d}n$$
$$\times \exp\left[\sum_x \left(-\frac{\beta}{16}\,\omega^2_{x\mu\nu} + \frac{\mathrm{i}}{2}\,\overline{b}_{x\mu\nu}\cdot\omega_{x\mu\nu} - \frac{\mathrm{i}}{2}\,\epsilon_{\rho\mu\nu}\overline{\nabla}_\rho\varphi\cdot\omega_{x\mu\nu} + 4\pi\mathrm{i}\,|\varphi_x|\,m_x \right) \right] \tag{57}$$

Here, $\overline{b}_{x\mu\nu}$ stands for the $\mathbb{R}^3$-valued 2-chain

$$\overline{b}_{x\mu\nu} = (j+1/2)\,n\,S_{x\mu\nu}\,, \tag{58}$$

i.e. $\overline{b}$ is proportional to the tensor product of the 2-chain $S_{x\mu\nu}$ (defining the surface S) and the unit vector n in the Lie algebra vector space $\mathbb{R}^3$. We can express it also as

$$\overline{b}_{x\mu\nu} = -u_\mu\,(u\cdot\nabla)^{-1} J_{x\nu} + u_\nu\,(u\cdot\nabla)^{-1} J_{x\mu} \tag{59}$$

where u is the unit vector in the x_1-direction and the current is defined by

$$J_{x\mu} := (j+1/2)\,n\,C_{x\mu}\,. \tag{60}$$

Next we decompactify $\omega_{x\mu\nu}$ and integrate over it:

$$\langle \mathrm{tr}_j W_C \rangle = \frac{1}{Z} \int\limits_{\mathbb{R}^3} \Bigg(\prod_x \mathrm{d}^3\varphi_x \Bigg) \sum_{m_x \in \mathbb{Z}} \frac{2j+1}{4\pi} \int\limits_{S^2} \mathrm{d}n$$
$$\times \exp\left[\sum_x \left(-\frac{2}{\beta}\,\big(\overline{\nabla}_\mu\varphi + \overline{b}_{x\mu}\big)^2 + 4\pi\mathrm{i}\,|\varphi_x|\,m_x \right) \right] \tag{61}$$

In this expression, we switched from the 2-chain $\overline{b}_{x\mu\nu}$ to the 1-chain

$$\overline{b}_{x\rho} = \frac{1}{2}\,\epsilon_{\rho\mu\nu}\overline{b}_{x\mu\nu}\,. \tag{62}$$

As in the abelian case, we factor off the Coulomb energy (the so-called spin-wave part) by making a change of variables and using the identity[3]

$$\nabla_\mu \overline{b}^a_{x\mu}\Delta^{-1}\nabla_\mu\overline{b}^a_{x\mu} + \overline{b}^2_{x\mu} = -J^a_{x\mu}\Delta^{-1}J^a_{x\mu}\,. \tag{63}$$

The final result reads

$$\langle \mathrm{tr}_j W_C \rangle = \frac{1}{Z} \int\limits_{\mathbb{R}^3} \Bigg(\prod_x \mathrm{d}^3\varphi_x \Bigg) \sum_{m_x \in \mathbb{Z}} \frac{2j+1}{4\pi} \int\limits_{S^2} \mathrm{d}n$$
$$\times \exp\left[\sum_x \left(\frac{2}{\beta}\,\varphi^a_x\Delta\varphi^a_x + 4\pi\mathrm{i}\,\big|\varphi_x + \Delta^{-1}\nabla_\mu\overline{b}_{x\mu}\big|\,m_x + \frac{2}{\beta}\,J^a_{x\mu}\Delta^{-1}J^a_{x\mu} \right) \right] \tag{64}$$

[2]Throughout it is assumed that we apply the same steps within the partition function Z by which we divide.

[3]Eq. (63) can be proven by starting from the expression $\epsilon_{\rho\mu\nu}\nabla_\mu\overline{b}^a_{x\mu}\Delta^{-1}\epsilon_{\rho\kappa\lambda}\nabla_\kappa\overline{b}^a_{x\lambda}$ and observing that J has no divergence.

It should be kept in mind that $\overline{b}_{x\mu\nu}$ depends on the direction of the unit vector n via eq. (58).

We propose (64) as a non-abelian generalization of the photon-monopole representation (4). Recall that the field φ in (4) is interpreted as a dual photon field that mediates the interaction between currents, between currents and monopoles, and among monopoles themselves. We call it dual, since it originates from the gauge potential of the $\mathbb{Z}$ gauge theory, which is dual to the original U(1) gauge theory. Similarly, we interpret the field φ in (64) as a dual gluon field:

- φ mediates the Coulomb interaction

$$V_{JJ} := -\frac{2}{\beta}\sum_{x} J^a_{x\mu}\Delta^{-1}J^a_{x\mu} = -\frac{1}{2}\,ag^2(j+1/2)^2\sum_{xy} C_{x\mu}\Delta^{-1}_{xy}C_{y\mu}\,, \quad j\neq 0\,, \tag{65}$$

  in (61). The latter agrees roughly[4] with the tree-level result of standard perturbation theory [20, 21]: there one would have

$$V^{\text{tree}}_{JJ} = -\frac{1}{2}\,ag^2 j(j+1)\sum_{xy} C_{x\mu}\Delta^{-1}_{xy}C_{y\mu}\,. \tag{66}$$

- φ has 3 degrees of freedom per point, which agrees with the fact that in 3 dimensions we have 1 physical degree of freedom per gluon and altogether 3 gluons for SU(2).

We interpret the m's as the non-abelian generalization of monopoles in 3-dimensional SU(2) lattice gauge theory.

5 Derivation of the area law?

The close similarity between (4) and (64) suggests that it could provide a way to generalize the derivation of confinement from compact 3d QED to compact 3d QCD with gauge group SU(2). The main difference between (4) and (64) consists in the fact that the gluon field φ is $\mathbb{R}^3$- and not $\mathbb{R}$-valued and that there appears a modulus in the second term. This renders the field theory nonlinear. As a result, we cannot perform the Gaussian integration as in (4) and write down the analogue of the Coulomb gas representation (9).

Instead we will proceed as suggested in the remark after equation (12). We will assume that a correct treatment has a similar effect as for U(1): that after renormalization down to a suitable cutoff scale M,

- the field φ is replaced by an effective field with a regulated Laplace operator, and
- a self-energy for the monopoles is generated.

[4] The reader may wonder why the formula gives a nonzero potential when j is zero. The answer is that we *do* get a zero Coulomb potential when the spin is zero from the start. If we use the Kirillov trace formula, however, and set $j = 0$ at the end of the derivation, the semiclassical approximation creates an error and a nonzero offset in the j-dependence.

If this is true, we will get

$$\langle \mathrm{tr}_j W_C \rangle = \frac{1}{Z} \int\limits_{\mathbb{R}^3} \left(\prod_x \mathrm{d}^3\varphi_x \right) \sum_{m_x \in \mathbb{Z}} \frac{2j+1}{4\pi} \int\limits_{S^2} \mathrm{d}n \\ \times \exp \left[\sum_x \left(\frac{2}{\beta}\, \varphi^a_x \tilde{\Delta} \varphi^a_x - 2\pi^2 \beta v_0\, m_x^2 + 4\pi \mathrm{i} \left| \varphi_x + \Delta^{-1} \nabla_\mu \overline{b}_{x\mu} \right| m_x + \frac{2}{\beta}\, J^a_{x\mu} \Delta^{-1} J^a_{x\mu} \right) \right] \tag{67}$$

where, as in section (2),

$$-\tilde{\Delta}^{-1} := -\Delta^{-1} - \left(-\Delta + M^2 \right)^{-1} . \tag{68}$$

and v_0 is some constant. As before, we abbreviate

$$\eta_x = 4\pi \Delta^{-1} \nabla_\mu \overline{b}_{x\mu} , \tag{69}$$

and apply a rescaling and shift on φ:

$$\langle \mathrm{tr}_j W_C \rangle = \frac{1}{Z} \int\limits_{\mathbb{R}^3} \left(\prod_x \mathrm{d}^3\varphi_x \right) \sum_{m_x} \frac{2j+1}{4\pi} \int\limits_{S^2} \mathrm{d}n \;\; \mathrm{e}^{-V_{JJ}} \\ \times \exp \left[\sum_x \left(\frac{1}{8\pi^2 \beta} (\varphi_x - \eta_x) \tilde{\Delta} (\varphi_x - \eta_x) - 2\pi^2 \beta v_0\, m_x^2 + \mathrm{i}\, m_x |\varphi_x| \right) \right] \tag{70}$$

Then, the dilute gas approximation yields

$$\langle \mathrm{tr}_j W_C \rangle = \frac{1}{Z} \int\limits_{\mathbb{R}^3} \left(\prod_x \mathrm{d}^3\varphi_x \right) \frac{2j+1}{4\pi} \int\limits_{S^2} \mathrm{d}n \;\; \mathrm{e}^{-V_{JJ}} \\ \times \exp \left[\sum_x \left(\frac{1}{8\pi^2 \beta} (\varphi_x - \eta_x) \tilde{\Delta} (\varphi_x - \eta_x) + 2\, \mathrm{e}^{-2\pi^2 \beta v_0} \cos |\varphi_x| \right) \right] . \tag{71}$$

Recall that

$$\overline{b}_{x\mu\nu} = (j + 1/2)\, n\, S_{x\mu\nu} , \tag{72}$$

and observe that the action is invariant under global rotations

$$\varphi'^a = R^a{}_c \varphi^c , \qquad \overline{b}'^a_{x\mu} = R^a{}_c \overline{b}^c_{x\mu} . \tag{73}$$

The saddle points are determined by the equation

$$-\tilde{\Delta} (\varphi - \eta)^a = M_D^2 \frac{\varphi^a}{|\varphi|} \sin |\varphi| \tag{74}$$

where

$$M_D^2 = 8\pi^2 \beta\, \mathrm{e}^{-2\pi^2 \beta v_0} . \tag{75}$$

We replace the regulated Laplacian by the full Laplacian, and plug in eq. (69):

$$\Delta \varphi^a = 4\pi \nabla_\mu \overline{b}^a_{x\mu} - M_D^2 \frac{\varphi^a}{|\varphi|} \sin |\varphi| \tag{76}$$

For simplicity, we assume now that $j = 1/2$. Consider first the equation in the region above or below the surface S:

$$\Delta\varphi^a = -M_D^2 \frac{\varphi^a}{|\varphi|} \sin|\varphi| \tag{77}$$

We can find a simple solution for this if we assume that the direction of φ is constant. Then the equation reduces to the nonlinear Debye equation for $|\varphi|$.

$$\Delta|\varphi| = -M_D^2 \sin|\varphi| \,, \tag{78}$$

We treat this as a quasi 1-dimensional problem and approximate it by the continuum equation

$$\frac{\partial^2|\varphi|}{\partial x_3^2} = -M_D^2 \sin|\varphi| \,. \tag{79}$$

The term $4\pi\nabla_\mu \overline{b}^a_{x\mu}$ in (76) is only nonzero at the surface S and gives

$$-\,4\pi\nabla_\mu \overline{b}^a_{x\mu} = 4\pi\,(\delta_{x_3,0} - \delta_{x_3,-a})\, n^a \,. \tag{80}$$

This implies that at S the field value φ has to make a jump by $4\pi\, n$.

If the jump was $2\pi\, n$, we could construct the solution as in the abelian case for charge $q = 1$. The fact that it is $4\pi\, n$ creates a slight (but harmless) complication, and can be treated like the doubly charged loop of U(1) [22].

Recall that we have a certain freedom in choosing the particular solution $\overline{b}_{x\mu\nu}$. Instead of using the minimal surface, so that

$$\overline{b} = n\, S \,, \tag{81}$$

we could take two surfaces S_+ and S_- s.t. $\partial S_+ = \partial S_- = C$, and set

$$\overline{b}_{x\mu\nu} = \frac{1}{2}\, n\, \left(S_+ + S_-\right) \,. \tag{82}$$

Now the φ-field has to jump two times by $2\pi\, n$: once along S_-, and a second time along S_+. Thus, we can treat the situation near each surface similarly as for charge $q = 1$.

Imagine that S_+ results from "stretching" the minimal surface S to an x_3-value $\overline{x}_3 > 0$, and that S_- is the mirror image of S_+ w.r.t. to the x_1-x_2-plane. Then, a solution to (76) is approximatively given by two domain walls: namely,

$$\varphi^{\rm cl}(x) \quad \approx \quad \begin{cases} 4\, n \, \arctan\left(\mathrm{e}^{-M_D(x_3+\overline{x}_3)}\right) - 2\pi \,, & (x_1,x_2) \in S \,, \quad x_3 < 0 \,, \\ 4\, n \, \arctan\left(\mathrm{e}^{-M_D(x_3-\overline{x}_3)}\right) \,, & (x_1,x_2) \in S \,, \quad x_3 > 0 \,, \\ 0 \,, & \text{otherwise} \,. \end{cases} \tag{83}$$

We plug this solution back into (71) and use the trivial saddle point for Z:

$$\langle \mathrm{tr}_j W_C \rangle = \frac{1}{4\pi} \int\limits_{S^2} \mathrm{d}n \;\; \mathrm{e}^{-V_{JJ}} \exp\left[\sum_x \left(\frac{1}{8\pi^2\beta}\, (\varphi^{\rm cl}_x - \eta_x) \tilde{\Delta} (\varphi^{\rm cl}_x - \eta_x) + 2\, \mathrm{e}^{-2\pi^2\beta v_0} \cos|\varphi^{\rm cl}_x| \right) \right] \tag{84}$$

The associated action equals $A\mathcal{S}_{\rm 2dw}$, where A is the area of the surface S and $\mathcal{S}_{\rm 2dw}$ is the action (or minus the energy) of the two domain walls, i.e.

$$\mathcal{S}_{\rm 2dw} \approx \frac{4M_D}{\pi^2\beta}\,. \tag{85}$$

The action does not depend on n, so the integral over n drops out. The result is an area law

$$\langle {\rm tr}_j W_C \rangle \quad \approx \quad 2\,\exp\left(-V_{JJ} - \sigma A\right) \tag{86}$$

with a string tension

$$\sigma = \frac{4M_D}{\pi^2\beta}\,. \tag{87}$$

As in the case of the Coulomb potential, it would be wrong to set j to zero in formula (72). This would yield a confining potential when there is actually no Wilson loop. The reason is again that we use the Kirillov trace formula in conjunction with semiclassical integrations, and that creates a wrong offset for the j-dependence. The correct procedure is to set $j = 0$ at the beginning of the derivation. Then, the resulting potential is zero, as it should.

6 Summary and discussion

In this paper, we have derived a gluon-monopole representation for SU(2) lattice gauge theory in dimension $d = 3$. We propose it as a generalization of the photon-monopole representation of Polyakov [2, 3] and Banks et al. [1].

Our derivation extends an earlier work by Borisenko, Voloshin and Faber [15]. By including a lowest-order contribution from a Wilson loop, we arrive at a gluon-monopole representation with a coupling to a source current. It produces approximatively the tree-level SU(2) Coulomb interaction, and the coupling between monopoles, dual gluons and current is similar to that of U(1). The difference is that the dual gluon field is $\mathbb{R}^3$-valued and monopoles couple to the *length* of field vectors. Thus, the coupling is nonlinear.

The analogy with U(1) suggests a possible derivation of confinement. It requires us, however, to make an additional assumption on the monopole self-energy. In the abelian case, one can go to the Coulomb gas representation, extract a monopole self-energy and transform back to an *effective* photon-monopole representation. Here, we cannot compute the analogue of a Coulomb gas, since the integral over the dual gluon field is non-Gaussian. Instead we *assumed* that a renormalization generates a monopole self-energy at a lower energy scale M. From there on, we can proceed as in Polykov's derivation and arrive at a non-vanishing string tension for the Wilson loop. Further investigation has to show if our heuristic assumption can be justified.

Since our approach applies to weak coupling, it is very different from the derivation by Karabali, Kim and Nair, which requires strong coupling [23, 24, 25]. The weak-coupling method of Orland assumes an anisotropic coupling [26, 27, 28].

Monopole-based scenarios of confinement have been criticized on the ground that they do not predict the observed Casimir scaling and N-ality dependence of the string tension [29, 30]. Can our gluon-monopole representation improve the situation and capture these genuinely non-abelian features?

According to the dilute gas and saddle point approximation of sec. 5, the problem could persist: we considered simple solutions, where the saddle point equation reduces to an equation for the length of the field vector. This equation is the same as the nonlinear Debye equation for U(1). This suggests that the string tension is proportional to the representation, as in the abelian case [22], and that color screening does not appear. The argument is not conclusive, however, since it included a heuristic step (the assumption on the monopole self-energy).

We also expect that improved derivations will produce additional features in the gluon-monopole representation that are not visible at this stage.

This is indicated by the derivation from the spin foam representation: it is the representation in which the strong-coupling expansion is performed, and in that limit color screening is very simple to understand. How does it carry over to the weak-coupling phase? We pointed out in paper II that our analysis yields additional solutions: they include the well-known tube-like diagrams that screen color at strong coupling. So far we have discarded these solutions for simplicity, but we suspect that they are the source of color screening at weak coupling.

Casimir scaling is another property that is evident when we look at spin foam sums at strong coupling. According to lattice simulations, this behaviour persists in the weak-coupling regime. If Casimir scaling gets lost in our approximations, it would be interesting to see how exactly that happens, and how one could improve the model, so that Casimir scaling is preserved.

The approach of this paper can been seen as a new type of semiclassical weak-coupling method that extracts information on non-trivial field configurations and their non-perturbative effects. It needs to be checked if it can be reliably applied at large quark distances, or if there occur problems due to higher-order corrections. How does the situation of 3d SU(2) compare to that of 3d U(1), where semiclassical techniques work well [2, 3]?

Can the derivation of this paper be extended to 4 dimensions? This depends on whether it is possible to construct an explicit plaquette representation in dimension 4.

At the end of paper I, we discuss and compare the results from all three papers.

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